

A Sharp upper bound for the spectral radius of a nonnegative matrix and applications*

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Dedicated to Professor Miroslav Fiedler on the occasion of his 90th birthday

Abstract In this paper, we obtain a sharp upper bound for the spectral radius of a nonnegative matrix. This result is used to present upper bounds for the adjacency spectral radius, the Laplacian spectral radius, the signless Laplacian spectral radius, the distance spectral radius, the distance Laplacian spectral radius, the distance signless Laplacian spectral radius of a graph or a digraph. These results are new or generalize some known results.

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1 Introduction

We begin by recalling some definitions. Let M be an $n \times n$ matrix, $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of M . It is obvious that the eigenvalues may be complex numbers since M is not symmetric in general. We usually assume that $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$. The spectral radius of M is defined as $\rho(M) = |\lambda_1|$, i.e., it is the largest modulus of the eigenvalues of M . If M is a nonnegative matrix, it follows from the Perron-Frobenius theorem that the spectral

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radius $\rho(M)$ is a eigenvalue of M . If M is a nonnegative irreducible matrix, it follows from the Perron-Frobenius theorem that $\rho(M) = \lambda_1$ is simple.

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G)$. The Laplacian matrix and the signless Laplacian matrix of G are defined as

$$L(G) = \text{diag}(G) - A(G), \quad Q(G) = \text{diag}(G) + A(G),$$

respectively, where $A(G) = (a_{ij})$ is the adjacency matrix of G , $\text{diag}(G) = \text{diag}(d_1, d_2, \dots, d_n)$ is the diagonal matrix of vertex degrees of G and d_i is the degree of vertex v_i . The spectral radius of $A(G)$, $L(G)$ and $Q(G)$, denoted by $\rho(G)$, $\mu(G)$ and $q(G)$, are called the (adjacency) spectral radius of G , the Laplacian spectral radius of G , and the signless Laplacian spectral radius of G , respectively. In 1973, Fiedler [9] studied the Laplacian spectra, in particular, the second small eigenvalue which is called algebra connectivity. Since then, the Laplacian matrix have been extensively investigated. Further, Fiedler [10] gave an excellent survey for the Laplacian matrix.

Let $G = (V, E)$ be a connected graph with vertex set $V = V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G)$. For $u, v \in V(G)$, the distance between u and v , denoted by $d_G(u, v)$, is the length of the shortest path connecting them in G . The distance matrix of G is the $n \times n$ matrix $\mathcal{D}(G) = (d_{ij})$ where $d_{ij} = d_G(v_i, v_j)$. In fact, for $1 \leq i \leq n$, the transmission of vertex v_i , $\text{Tr}_G(v_i)$ is just the i -th row sum of $\mathcal{D}(G)$. So for convenience, we also call $\text{Tr}_G(v_i)$ the distance degree of vertex v_i in G , denoted by D_i , that is, $D_i = \sum_{j=1}^n d_{ij} = \text{Tr}_G(v_i)$.

Let $\text{Tr}(G) = \text{diag}(D_1, D_2, \dots, D_n)$ be the diagonal matrix of vertex transmissions of G . The distance Laplacian matrix and the distance signless Laplacian matrix of G are the $n \times n$ matrix defined by Aouchiche and Hansen as [1]

$$\mathcal{L}(G) = \text{Tr}(G) - \mathcal{D}(G), \quad \mathcal{Q}(G) = \text{Tr}(G) + \mathcal{D}(G).$$

The spectral radius of $\mathcal{D}(G)$, $\mathcal{L}(G)$ and $\mathcal{Q}(G)$, denoted by $\rho^{\mathcal{D}}(G)$, $\mu^{\mathcal{D}}(G)$ and $q^{\mathcal{D}}(G)$, are called the distance spectral radius of G , the distance Laplacian spectral radius of G , and the distance signless Laplacian spectral radius of G , respectively.

Let $\vec{G} = (V, E)$ be a digraph, where $V = V(\vec{G}) = \{v_1, v_2, \dots, v_n\}$ and $E = E(\vec{G})$ are the vertex set and arc set of \vec{G} , respectively. A digraph \vec{G} is simple if it has no loops and multiple arcs. A digraph \vec{G} is strongly connected if for every pair of vertices $v_i, v_j \in V(\vec{G})$, there are directed paths from v_i to v_j and from v_j to v_i . In this paper, we consider finite,

simple digraphs.

Let \vec{G} be a digraph. Let $N_{\vec{G}}^+(v_i) = \{v_j \in V(\vec{G}) \mid (v_i, v_j) \in E(\vec{G})\}$ denote the set of the out-neighbors of v_i , $d_i^+ = |N_{\vec{G}}^+(v_i)|$ denote the out-degree of the vertex v_i in \vec{G} .

For a digraph \vec{G} , let $A(\vec{G}) = (a_{ij})$ denote the adjacency matrix of \vec{G} , where a_{ij} is equal to the number of arcs (v_i, v_j) . Let $diag(\vec{G}) = diag(d_1^+, d_2^+, \dots, d_n^+)$ be the diagonal matrix of vertex out-degrees of \vec{G} and

$$L(\vec{G}) = diag(\vec{G}) - A(\vec{G}), \quad Q(\vec{G}) = diag(\vec{G}) + A(\vec{G})$$

be the Laplacian matrix of \vec{G} and the signless Laplacian matrix of \vec{G} , respectively. The spectral radius of $A(\vec{G})$, $L(\vec{G})$ and $Q(\vec{G})$, denoted by $\rho(\vec{G})$, $\mu(\vec{G})$ and $q(\vec{G})$, are called the (adjacency) spectral radius of \vec{G} , the Laplacian spectral radius of \vec{G} , and the signless Laplacian spectral radius of \vec{G} , respectively.

For $u, v \in V(\vec{G})$, the distance from u to v , denoted by $d_{\vec{G}}(u, v)$, is the length of the shortest directed path from u to v in \vec{G} . For $u \in V(\vec{G})$, the transmission of vertex u in \vec{G} is the sum of distances from u to all other vertices of \vec{G} , denoted by $Tr_{\vec{G}}(u)$.

Let \vec{G} be a strong connected digraph with vertex set $V(\vec{G}) = \{v_1, v_2, \dots, v_n\}$. The distance matrix of \vec{G} is the $n \times n$ matrix $\mathcal{D}(\vec{G}) = (d_{ij})$ where $d_{ij} = d_{\vec{G}}(v_i, v_j)$. In fact, for $1 \leq i \leq n$, the transmission of vertex v_i , $Tr_{\vec{G}}(v_i)$ is just the i -th row sum of $\mathcal{D}(\vec{G})$. So for convenience, we also call $Tr_{\vec{G}}(v_i)$ the distance degree of vertex v_i in \vec{G} , denoted by D_i^+ , that is, $D_i^+ = \sum_{j=1}^n d_{ij} = Tr_{\vec{G}}(v_i)$.

Let $Tr(\vec{G}) = diag(D_1^+, D_2^+, \dots, D_n^+)$ be the diagonal matrix of vertex transmissions of \vec{G} . The distance Laplacian matrix and the distance signless Laplacian matrix of \vec{G} are the $n \times n$ matrices defined similar to the undirected graph by Aouchiche and Hansen as ([1])

$$\mathcal{L}(\vec{G}) = Tr(\vec{G}) - \mathcal{D}(\vec{G}), \quad \mathcal{Q}(\vec{G}) = Tr(\vec{G}) + \mathcal{D}(\vec{G}).$$

The spectral radius of $\mathcal{D}(\vec{G})$, $\mathcal{L}(\vec{G})$ and $\mathcal{Q}(\vec{G})$, denoted by $\rho^{\mathcal{D}}(\vec{G})$, $\mu^{\mathcal{D}}(\vec{G})$ and $q^{\mathcal{D}}(\vec{G})$, are called the distance spectral radius of \vec{G} , the distance Laplacian spectral radius of \vec{G} and the distance signless Laplacian spectral radius of \vec{G} , respectively.

Let $G = (V, E)$ be a graph, for $v_i, v_j \in V$, if v_i is adjacent to v_j , we denote it by $i \sim j$. Moreover, we call $m_i = \frac{\sum_{i \sim j} d_j}{d_i}$ the average degree of the neighbors of v_i . Let $\vec{G} = (V, E)$ be a digraph, for $v_i, v_j \in V$, if arc $(v_i, v_j) \in E$, we denote it by $i \sim j$. Moreover, we call

$m_i^+ = \frac{\sum_{j \sim i} d_j^+}{d_i^+}$ the average out-degree of the out-neighbors of v_i , where d_i^+ is the out-degree of vertex v_i in \vec{G} .

A regular graph is a graph where every vertex has the same degree. A *bipartite semi-regular* graph is a bipartite graph $G = (U, V, E)$ for which every two vertices on the same side of the given bipartition have the same degree as each other.

So far, there are many results on the bounds of the spectral radius of a matrix and a nonnegative matrix, the spectral radius, the Laplacian spectral radius, the signless Laplacian spectral radius, the distance spectral radius, the distance Laplacian spectral radius and the distance signless Laplacian spectral radius of a graph and a digraph, see [1,3–13,15–17]. The following are some results on the above spectral radii of graphs and digraphs in terms of degree, average degree, out-degree and so on.

$$\rho(G) \leq \max_{1 \leq i \leq n} \{ \sqrt{d_i m_i} \} \quad (1.1)$$

$$\mu(G) \leq \max_{1 \leq i \leq n} \{ d_i + \sqrt{d_i m_i} \} \quad (1.2)$$

$$q(G) \leq \max_{1 \leq i \leq n} \{ d_i + \sqrt{d_i m_i} \} \quad (1.3)$$

$$q(\vec{G}) \leq \max_{1 \leq i \leq n} \left\{ d_i^+ + \sqrt{\sum_{j \sim i} d_j^+} \right\} \quad (1.4)$$

We can see that there are few results about the distance Laplacian spectral radius of G , the Laplacian spectral radius of \vec{G} , the distance Laplacian spectral radius of \vec{G} and the distance signless Laplacian spectral radius of \vec{G} . Maybe one reason is the Laplacian matrix and the distance Laplacian matrix are not nonnegative matrices.

In this paper, we obtain sharp upper bound for the spectral radius of a matrix or a non-negative matrix in Section 2, and then we apply these bounds to various matrices associated with a graph or a digraph, obtain some new results or known results about various spectral radii, including the (adjacency) spectral radius, the Laplacian spectral radius, the signless Laplacian spectral radius, the distance spectral radius, the distance Laplacian spectral radius, the distance signless Laplacian spectral radius and so on.

2 Main results

In this section, we will obtain the sharp upper bound for the spectral radius of a (non-negative) matrix, The techniques used in this section is motivated by [21] et al.

Theorem 2.1. *Let $B = (b_{ij})$ be an $n \times n$ nonnegative matrix, l_i be the number of the*

nonzero entries except for the diagonal entry of the i th row for any $i \in \{1, 2, \dots, n\}$, say, $l_i = |\{b_{ij} \mid b_{ij} \neq 0, j \in \{1, 2, \dots, n\} \setminus \{i\}\}|$, $X = (x_1, x_2, \dots, x_n)^T$ be the eigenvector of B corresponding to the eigenvalue $\rho(B)$. Then

$$\rho(B) \leq \max_{1 \leq i \leq n} \left\{ b_{ii} + \sqrt{\sum_{k=1, k \neq i}^n l_k b_{ki}^2} \right\}. \quad (2.1)$$

Moreover, if the equality in (2.1) holds, then $b_{ii} + \sqrt{\sum_{k=1, k \neq i}^n l_k b_{ki}^2} = b_{jj} + \sqrt{\sum_{k=1, k \neq j}^n l_k b_{kj}^2}$ for any $i, j \in \{s \mid x_s \neq 0, 1 \leq s \leq n\}$. Furthermore, if B is irreducible, and the equality in (2.1) holds, then $b_{ii} + \sqrt{\sum_{k=1, k \neq i}^n l_k b_{ki}^2} = b_{jj} + \sqrt{\sum_{k=1, k \neq j}^n l_k b_{kj}^2}$ for any $i, j \in \{1, 2, \dots, n\}$.

Proof. For each $i \in \{1, 2, \dots, n\}$, by $BX = \rho(B)X$, we have $\rho(B)x_i = \sum_{j=1}^n b_{ij}x_j$, then

$$(\rho(B) - b_{ii})x_i = \sum_{j \neq i, b_{ij} \neq 0} b_{ij}x_j,$$

and thus by Cauchy inequality, we have

$$(\rho(B) - b_{ii})^2 x_i^2 = \left(\sum_{j \neq i, b_{ij} \neq 0} b_{ij}x_j \right)^2 \leq l_i \sum_{j \neq i, b_{ij} \neq 0} (b_{ij}x_j)^2.$$

Then

$$\sum_{i=1}^n [(\rho(B) - b_{ii})x_i]^2 \leq \sum_{i=1}^n \left(l_i \sum_{j \neq i, b_{ij} \neq 0} (b_{ij}x_j)^2 \right) = \sum_{i=1}^n \left(\sum_{j \neq i, b_{ij} \neq 0} l_i b_{ij}^2 x_j^2 \right) = \sum_{i=1}^n \left[\left(\sum_{k=1, k \neq i}^n l_k b_{ki}^2 \right) x_i^2 \right],$$

thus we have

$$\sum_{i=1}^n \left((\rho(B) - b_{ii})^2 - \sum_{k=1, k \neq i}^n l_k b_{ki}^2 \right) x_i^2 \leq 0. \quad (2.2)$$

Therefore there must exist some $j \in \{1, 2, \dots, n\}$ such that

$$(\rho(B) - b_{jj})^2 - \sum_{k \neq j} l_k b_{kj}^2 \leq 0,$$

so

$$\rho(B) \leq b_{jj} + \sqrt{\sum_{k \neq j} l_k b_{kj}^2} \leq \max_{1 \leq i \leq n} \left\{ b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} \right\}.$$

If $\rho(B) = \max_{1 \leq i \leq n} \left\{ b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} \right\}$, then for any $j \in \{1, 2, \dots, n\}$, we have $\rho(B) \geq b_{jj} + \sqrt{\sum_{k \neq j} l_k b_{kj}^2}$, then

$$(\rho(B) - b_{jj})^2 - \sum_{k \neq j} l_k b_{kj}^2 \geq 0, \quad (2.3)$$

and thus

$$\sum_{j=1}^n ((\rho(B) - b_{jj})^2 - \sum_{k \neq j} l_k b_{kj}^2) x_j^2 \geq 0. \quad (2.4)$$

Combining (2.2) and (2.4), it implies that

$$\sum_{i=1}^n ((\rho(B) - b_{ii})^2 - \sum_{k \neq i} l_k b_{ki}^2) x_i^2 = 0.$$

Noting that (2.3) holds for any $j \in \{1, 2, \dots, n\}$, we have $(\rho(B) - b_{ii})^2 - \sum_{k \neq i} l_k b_{ki}^2 = 0$ for any $i \in \{s \mid x_s \neq 0, 1 \leq s \leq n\}$, and thus $b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} = b_{jj} + \sqrt{\sum_{k \neq j} l_k b_{kj}^2}$ for any $i, j \in \{s \mid x_s \neq 0, 1 \leq s \leq n\}$.

Furthermore, if B is irreducible, then $x_i > 0$ for each $i \in \{1, 2, \dots, n\}$ by Perron-Frobenius theorem, and thus $b_{ii} + \sqrt{\sum_{k=1, k \neq i}^n l_k b_{ki}^2} = b_{jj} + \sqrt{\sum_{k=1, k \neq j}^n l_k b_{kj}^2}$ for any $i, j \in \{1, 2, \dots, n\}$ if the equality in (2.1) holds. \square

It is natural that we want to know under what conditions the equality in (2.1) holds.

Question 2.2. *Look for the necessity and sufficiency conditions of the equality in (2.1) holds.*

Lemma 2.3. *([14]) Let $B = (b_{ij})$ be an $n \times n$ nonnegative irreducible matrix and $A = (a_{ij})$ is a complex matrix. Let $|A| = (|a_{ij}|)$, if $b_{ij} \geq |a_{ij}|$ for any $i, j \in \{1, 2, \dots, n\}$, we denote by $B \geq |A|$, then $\rho(B) \geq \rho(A)$.*

By Lemma 2.3 we know that for any connected graph G and any strong connected digraph \vec{G} , $\mu(G) \leq q(G)$ and $\mu(\vec{G}) \leq q(\vec{G})$. In fact, we have

Lemma 2.4. *([2]) Let $G = (V, E)$ be a connected graph on n vertices. Then $\mu(G) \leq q(G)$, with equality if and only if G is a bipartite graph.*

Corollary 2.5. *Let $A = (a_{ij})$ be an $n \times n$ complex irreducible matrix, l_i be the number of the nonzero entries except for the diagonal entry of the i th row for any $i \in \{1, 2, \dots, n\}$, say,*

$l_i = |\{a_{ij} \mid a_{ij} \neq 0, j \in \{1, 2, \dots, n\} \setminus \{i\}\}|$. Then

$$\rho(A) \leq \max_{1 \leq i \leq n} \left\{ |a_{ii}| + \sqrt{\sum_{k=1, k \neq i}^n l_k |a_{ki}|^2} \right\}. \quad (2.5)$$

If the equality holds, then $|a_{ii}| + \sqrt{\sum_{k=1, k \neq i}^n l_k |a_{ki}|^2} = |a_{jj}| + \sqrt{\sum_{k=1, k \neq j}^n l_k |a_{kj}|^2}$ for any $i, j \in \{1, 2, \dots, n\}$.

Proof. Let $B = (|a_{ij}|)$, then B is a nonnegative irreducible matrix. Thus $\rho(A) \leq \rho(B)$ by Lemma 2.3, and (2.5) holds by Theorem 2.1. \square

3 Various spectral radii of a graph

Let G be a graph, the adjacency matrix $A(G)$, the Laplacian matrix $L(G)$, the signless Laplacian matrix $Q(G)$, and the (adjacency) spectral radius $\rho(G)$, the Laplacian spectral radius $\mu(G)$, the signless Laplacian spectral radius $q(G)$ are defined as Section 1. Let G be a connected graph, the distance matrix $\mathcal{D}(G)$, the distance Laplacian matrix $\mathcal{L}(G)$, the distance signless Laplacian matrix $\mathcal{Q}(G)$, and the distance spectral radius $\rho^{\mathcal{D}}(G)$, the distance Laplacian spectral radius $\mu^{\mathcal{D}}(G)$, the distance signless Laplacian spectral radius $q^{\mathcal{D}}(G)$ are defined as Section 1. In this section, we will apply Theorems 2.1 to $A(G)$, $Q(G)$, $\mathcal{D}(G)$ and $\mathcal{Q}(G)$, and apply Corollary 2.5 to $L(G)$ and $\mathcal{L}(G)$, to obtain some new results or known results on the spectral radius.

3.1 Adjacency spectral radius of a graph

Lemma 3.1. *Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$. For any $v_i \in V$, the degree of v_i and the average degree of the vertices adjacent to v_i are denoted by d_i and m_i , respectively. Then $d_1 m_1 = d_2 m_2 = \dots = d_n m_n$ holds if and only if G is a regular graph or a bipartite semi-regular graph.*

Proof. If G is a regular graph or a bipartite semi-regular graph, we can check that $d_1 m_1 = d_2 m_2 = \dots = d_n m_n$ holds immediately.

Conversely, let $d_1 m_1 = d_2 m_2 = \dots = d_n m_n$ holds. Now we show that G is a regular or a bipartite semi-regular graph.

Let vertex v_n be the lowest degree vertex in G , say, $d_n = \min\{d_i \mid 1 \leq i \leq n\}$. Let $d_n = r$, and the neighbors of v_n be $v_{i_1}, v_{i_2}, \dots, v_{i_r}$. Let $d_{i_1} = \max\{d_{i_j} \mid 1 \leq j \leq r\}$, denoted by $s = d_{i_1}$. Then $m_n \leq s$ by $m_i = \frac{\sum_{j \sim i} d_j}{d_i}$ and thus $d_n m_n = r m_n \leq r s$.

On the other hand, for vertex v_{i_1} , we have $d_{i_1} m_{i_1} = s m_{i_1} \geq r s$, then $r s \leq d_{i_1} m_{i_1} = d_n m_n \leq r s$, thus $d_{i_1} m_{i_1} = d_n m_n = r s$, it implies $m_n = s$ and $m_{i_1} = r$. Therefore by the definitions of s and r , we know $v_{i_1}, v_{i_2}, \dots, v_{i_r}$ must have the same degree, say, $s = d_{i_1} = d_{i_2} = \dots = d_{i_r}$, and all the neighbors of v_{i_1} must have the same degree r .

Similar to the above arguments, we can show that the vertices with degree r are adjacent to the vertices with degree s , and the vertices with degree s are adjacent to the vertices with degree r in G .

Now we assume that G is not bipartite. Then G has at least an odd cycle. Let $C = v_{j_1} v_{j_2} \dots v_{j_{2k-1}} v_{j_{2k}} v_{j_{2k+1}} v_{j_1}$ be an odd cycle of length $2k + 1$ in G . Clearly, the degree of the vertex v_{j_1} , say d_{j_1} , is either r or s . Without loss of generality, we assume that $d_{j_1} = r$. Since the vertices with degree r are adjacent to the vertices with degree s , and the vertices with degree s are adjacent to the vertices with degree r , then $d_{j_2} = s, d_{j_3} = r, \dots, d_{j_{2k-1}} = r, d_{j_{2k}} = s, d_{j_{2k+1}} = r$ and $d_{j_1} = s$ by $v_{j_{2k+1}}$ and v_{j_1} are adjacent, thus $r = s$, it implies that G is regular.

Hence the graph G is a regular graph or a bipartite semi-regular graph. \square

Theorem 3.2. ([4], Theorem 1.) *Let $G = (V, E)$ be a simple graph on n vertices. Then $\rho(G) \leq \max_{1 \leq i \leq n} \sqrt{d_i m_i}$. Moreover, if G is a connected graph, then the equality holds if and only if G is a regular or bipartite semi-regular graph.*

Proof. We apply Theorem 2.1 to $A(G)$. Since $b_{ii} = 0, b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise,} \end{cases}$ and $l_i = d_i$ for $i = 1, 2, \dots, n$, then $b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} = \sqrt{d_i m_i}$ for $i = 1, 2, \dots, n$, and thus $\rho(G) \leq \max_{1 \leq i \leq n} \sqrt{d_i m_i}$ by (2.1).

If G is a connected graph, now we show the equality holds if and only if G is a regular or bipartite semi-regular graph.

If G is a connected graph and $\rho(G) = \max_{1 \leq i \leq n} \sqrt{d_i m_i}$, then $\sqrt{d_1 m_1} = \sqrt{d_2 m_2} = \dots = \sqrt{d_n m_n}$ by Theorem 2.1, and thus $d_1 m_1 = d_2 m_2 = \dots = d_n m_n$. Therefore, G is a regular or bipartite semi-regular graph by Lemma 3.1.

On the other hand, if G is connected and G is a regular or bipartite semi-regular graph, then $d_1 m_1 = d_2 m_2 = \dots = d_n m_n$ by Lemma 3.1, thus $\sqrt{d_1 m_1} = \sqrt{d_2 m_2} = \dots = \sqrt{d_n m_n}$, and

$\rho(G) \leq \max_{1 \leq i \leq n} \sqrt{d_i m_i} = \sqrt{d_1 m_1}$. Then we complete the proof by the following two cases.

Case 1: G is a regular graph with degree r .

It is well known that $r = \sqrt{d_1 m_1}$ is an eigenvalue of G , so $\sqrt{d_1 m_1} \leq \rho(G)$. Thus $\rho(G) = \max_{1 \leq i \leq n} \sqrt{d_i m_i} = r$ by $\rho(G) \leq \sqrt{d_1 m_1} = r$.

Case 2: G is a bipartite semi-regular graph.

We assume that the two bipartition of G with degree r and s , respectively. It is easy to check that $\sqrt{rs} = \sqrt{d_1 m_1}$ is an eigenvalue of G , so $\sqrt{rs} \leq \rho(G)$. Thus $\rho(G) = \max_{1 \leq i \leq n} \sqrt{d_i m_i} = \sqrt{rs}$ by $\rho(G) \leq \sqrt{d_1 m_1} = \sqrt{rs}$. \square

3.2 (Signless) Laplacian spectral radius of a graph

Lemma 3.3. ([14]) *Let A be a nonnegative matrix with the spectral radius $\rho(A)$ and the row sum r_1, r_2, \dots, r_n . Then $\min_{1 \leq i \leq n} r_i \leq \rho(A) \leq \max_{1 \leq i \leq n} r_i$. Moreover, if A is a irreducible matrix, the one of equalities holds if and only if the row sums of A are all equal.*

Lemma 3.4. *Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$. For any $v_i \in V$, the degree of v_i and the average degree of the vertices adjacent to v_i are denoted by d_i and m_i , respectively. Then $d_1 + \sqrt{d_1 m_1} = d_2 + \sqrt{d_2 m_2} = \dots = d_n + \sqrt{d_n m_n}$ holds if and only if G is a regular graph.*

Proof. If G is a regular graph, we can check $d_1 + \sqrt{d_1 m_1} = d_2 + \sqrt{d_2 m_2} = \dots = d_n + \sqrt{d_n m_n}$ holds immediately.

Conversely, let $d_1 + \sqrt{d_1 m_1} = d_2 + \sqrt{d_2 m_2} = \dots = d_n + \sqrt{d_n m_n}$ holds. Now we show G is a regular graph.

Let vertex v_n be the lowest degree vertex in G , say, $d_n = \min\{d_i \mid 1 \leq i \leq n\}$. Let $d_n = r$, and the neighbors of v_n be $v_{i_1}, v_{i_2}, \dots, v_{i_r}$. Let $d_{i_1} = \max\{d_{i_j} \mid 1 \leq j \leq r\}$, denoted by $s = d_{i_1}$. It is obvious that $r \leq s$, $m_n \leq s$ by $m_i = \frac{\sum_{j \sim i} d_j}{d_i}$, and thus $d_n + \sqrt{d_n m_n} \leq r + \sqrt{rs}$.

On the other hand, for vertex v_{i_1} , we have $d_{i_1} m_{i_1} = s m_{i_1} \geq r s$, then $d_{i_1} + \sqrt{d_{i_1} m_{i_1}} \geq s + \sqrt{rs}$. Thus $s + \sqrt{rs} \leq d_{i_1} + \sqrt{d_{i_1} m_{i_1}} = d_n + \sqrt{d_n m_n} \leq r + \sqrt{rs} \leq s + \sqrt{rs}$, it implies $m_n = s$, $m_{i_1} = r$ and $r = s$. Therefore by the definitions of s and r , we know $v_{i_1}, v_{i_2}, \dots, v_{i_r}$ must have the same degree, say, $s = d_{i_1} = d_{i_2} = \dots = d_{i_r}$, and all the neighbors of v_{i_1} must have the same degree $r (= s)$.

Similar to the above arguments, we can show that the vertices with degree r are adjacent to the vertices with degree s , and the vertices with degree s are adjacent to the vertices with degree r in G . Then G is a regular graph by $r = s$. \square

Theorem 3.5. *Let $G = (V, E)$ be a simple graph on n vertices. Then*

(i)[21] $q(G) \leq \max_{1 \leq i \leq n} \{d_i + \sqrt{d_i m_i}\}$. Moreover, if G is a connected graph, the equality holds if and only if G is a regular graph.

(ii)[25] If G is a connected graph, then $\mu(G) \leq \max_{1 \leq i \leq n} \{d_i + \sqrt{d_i m_i}\}$, and the equality holds if and only if G is a bipartite regular graph.

Proof. Firstly, we show (i) holds.

We apply Theorem 2.1 to $Q(G)$. Since $b_{ii} = d_i$, $b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise,} \end{cases}$
 $l_i = d_i$ for $i = 1, 2, \dots, n$, then we have $b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} = d_i + \sqrt{d_i m_i}$ for $i = 1, 2, \dots, n$, and thus $q(G) \leq \max_{1 \leq i \leq n} \{d_i + \sqrt{d_i m_i}\}$ by (2.1).

Now we show if G is a connected graph, then the equality holds if and only if G is regular.

If G is a connected graph and $q(G) = \max_{1 \leq i \leq n} \{d_i + \sqrt{d_i m_i}\}$, then $d_1 + \sqrt{d_1 m_1} = d_2 + \sqrt{d_2 m_2} = \dots = d_n + \sqrt{d_n m_n}$ by Theorem 2.1, and thus G is a regular graph by Lemma 3.4.

On the other hand, if G is connected and G is a regular graph with degree r , then $d_1 + \sqrt{d_1 m_1} = d_2 + \sqrt{d_2 m_2} = \dots = d_n + \sqrt{d_n m_n} = 2r$ by Lemma 3.4 and $\max_{1 \leq i \leq n} \{d_i + \sqrt{d_i m_i}\} = 2r$. It is well known that $q(G) = 2r$ by Lemma 3.3, so $q(G) = \max_{1 \leq i \leq n} \{d_i + \sqrt{d_i m_i}\}$.

Similar to the proof of (i), by Corollary 2.5, Lemma 2.4 and the result of (i), we can show (ii) immediately, so we omit it. \square

3.3 Distance spectral radius of a graph

Theorem 3.6. *Let $G = (V, E)$ be a connected graph on n vertices. Then*

$$\rho^D(G) \leq \max_{1 \leq i \leq n} \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2}. \quad (3.1)$$

If the equality holds, then $\sum_{k=1}^n d_{ki}^2 = \sum_{k=1}^n d_{kj}^2$ for any $i, j \in \{1, 2, \dots, n\}$.

Proof. We apply Theorem 2.1 to $\mathcal{D}(G)$. Since $b_{ii} = d_{ii} = 0$, $b_{ij} = d_{ij} \neq 0$ when $i \neq j$ and $l_i = n-1$ for $i = 1, 2, \dots, n$, then $b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} = \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2}$ for $i = 1, 2, \dots, n$, and thus (3.1) holds by (2.1).

It is obvious that if the equality holds, then $\sum_{k=1}^n d_{ki}^2 = \sum_{k=1}^n d_{kj}^2$ for any $i, j = 1, 2, \dots, n$ by Theorem 2.1. \square

3.4 Distance (signless) Laplacian spectral radius of a graph

Theorem 3.7. *Let $G = (V, E)$ be a connected graph on n vertices. Then*

$$q^D(G) \leq \max_{1 \leq i \leq n} \left\{ D_i + \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2} \right\}, \quad (3.2)$$

and

$$\mu^D(G) \leq \max_{1 \leq i \leq n} \left\{ D_i + \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2} \right\}. \quad (3.3)$$

Moreover, if the equality in (3.2) (or (3.3)) holds, then $D_i + \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2} = D_j + \sqrt{(n-1) \sum_{k=1}^n d_{kj}^2}$ for any $i, j \in \{1, 2, \dots, n\}$.

Proof. We apply Theorem 2.1 to $\mathcal{Q}(G)$. Since $b_{ii} = D_i$, $b_{ij} = d_{ij}$ where $i \neq j$, and $l_i = n-1$ for $i = 1, 2, \dots, n$, then $b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} = D_i + \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2}$ for $i = 1, 2, \dots, n$, and thus (3.2) holds.

Similarly, we apply Corollary 2.5 to $\mathcal{L}(G)$ and (3.3) holds. \square

4 Various spectral radii of a digraph

Let \vec{G} be a connected digraph, the adjacency matrix $A(\vec{G})$, the Laplacian matrix $L(\vec{G})$, the signless Laplacian matrix $Q(\vec{G})$, and the (adjacency) spectral radius $\rho(\vec{G})$, the Laplacian spectral radius $\mu(\vec{G})$, the signless Laplacian spectral radius $q(\vec{G})$ are defined as Section 1. Let \vec{G} be a connected digraph, the distance matrix $\mathcal{D}(\vec{G})$, the distance Laplacian matrix $\mathcal{L}(\vec{G})$, the distance signless Laplacian matrix $\mathcal{Q}(\vec{G})$, and the distance spectral radius $\rho^D(\vec{G})$, the distance Laplacian spectral radius $\mu^D(\vec{G})$, the distance signless Laplacian spectral radius $q^D(\vec{G})$ are defined as Section 1. In this section, we will apply Theorems 2.1 to $A(\vec{G})$, $Q(\vec{G})$, $\mathcal{D}(\vec{G})$ and $\mathcal{Q}(\vec{G})$, and apply Corollary 2.5 to $L(\vec{G})$ and $\mathcal{L}(\vec{G})$, to obtain some new results or known results on the spectral radius.

4.1 Adjacency spectral radius of a digraph

Theorem 4.1. ([26], Corollary 3.2) Let $\vec{G} = (V, E)$ be a digraph on n vertices. Then

$$\rho(\vec{G}) \leq \max_{1 \leq i \leq n} \sqrt{\sum_{k \sim i} d_k^+}.$$

If \vec{G} is connected and the equality holds, then $\sum_{k \sim 1} d_k^+ = \sum_{k \sim 2} d_k^+ = \dots = \sum_{k \sim n} d_k^+$.

Proof. We apply Theorem 2.1 to $A(\vec{G})$. Since $b_{ii} = 0$, $b_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E; \\ 0, & \text{otherwise,} \end{cases}$ and $l_i = d_i^+$ for $i = 1, 2, \dots, n$, then $b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} = \sqrt{\sum_{k \sim i} d_k^+}$, and thus $\rho(\vec{G}) \leq \max_{1 \leq i \leq n} \sqrt{\sum_{k \sim i} d_k^+}$ by (2.1).

It is obvious that if \vec{G} is connected and the equality holds, then $\sum_{k \sim 1} d_k^+ = \sum_{k \sim 2} d_k^+ = \dots = \sum_{k \sim n} d_k^+$ by Theorem 2.1. \square

4.2 (Signless) Laplacian spectral radius of a digraph

Theorem 4.2. Let $\vec{G} = (V, E)$ be a digraph on n vertices. Then

- (i) ([3], Theorem 3.3) $q(\vec{G}) \leq \max_{1 \leq i \leq n} \{d_i^+ + \sqrt{\sum_{j \sim i} d_j^+}\}$. Moreover, if \vec{G} is connected and the equality holds, then $d_1^+ + \sqrt{\sum_{j \sim 1} d_j^+} = d_2^+ + \sqrt{\sum_{j \sim 2} d_j^+} = \dots = d_n^+ + \sqrt{\sum_{j \sim n} d_j^+}$.
- (ii) If \vec{G} is connected, then $\mu(\vec{G}) \leq \max_{1 \leq i \leq n} \{d_i^+ + \sqrt{\sum_{j \sim i} d_j^+}\}$, and if the equality holds, then $d_1^+ + \sqrt{\sum_{j \sim 1} d_j^+} = d_2^+ + \sqrt{\sum_{j \sim 2} d_j^+} = \dots = d_n^+ + \sqrt{\sum_{j \sim n} d_j^+}$.

Proof. We apply Theorem 2.1 to $Q(\vec{G})$. Since $b_{ii} = d_i^+$, $b_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E; \\ 0, & \text{otherwise,} \end{cases}$ and $l_i = d_i^+$ for $i = 1, 2, \dots, n$, then $b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} = d_i^+ + \sqrt{\sum_{j \sim i} d_j^+}$ for $i = 1, 2, \dots, n$, and thus $q(\vec{G}) \leq \max_{1 \leq i \leq n} \{d_i^+ + \sqrt{\sum_{j \sim i} d_j^+}\}$ by (2.1).

It is obvious that if \vec{G} is connected and the equality holds then $d_1^+ + \sqrt{\sum_{j \sim 1} d_j^+} = d_2^+ + \sqrt{\sum_{j \sim 2} d_j^+} = \dots = d_n^+ + \sqrt{\sum_{j \sim n} d_j^+}$ by Theorem 2.1.

Similar to the proof of (i), we can show (ii) immediately by Corollary 2.5, so we omit it. \square

4.3 Distance spectral radius of a digraph

Theorem 4.3. Let $\vec{G} = (V, E)$ be a strong connected digraph on n vertices. Then

$$\rho^D(\vec{G}) \leq \max_{1 \leq i \leq n} \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2}. \quad (4.1)$$

If the equality holds, then $\sum_{k=1}^n d_{ki}^2 = \sum_{k=1}^n d_{kj}^2$ for any $i, j \in \{1, 2, \dots, n\}$.

Proof. We apply Theorem 2.1 to $\mathcal{D}(\vec{G})$. Since $b_{ii} = d_{ii} = 0$, $b_{ij} = d_{ij} \neq 0$, and $l_i = n - 1$ for any $i = 1, 2, \dots, n$, then $b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} = \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2}$ for $i = 1, 2, \dots, n$, and thus (4.1) holds by (2.1).

It is easy that if the equality holds, then $\sum_{k=1}^n d_{ki}^2 = \sum_{k=1}^n d_{kj}^2$ for any $i, j \in \{1, 2, \dots, n\}$. \square

4.4 Distance (signless) Laplacian spectral radius of a digraph

Theorem 4.4. Let $\vec{G} = (V, E)$ be a strong connected digraph on n vertices. Then

$$q^D(\vec{G}) \leq \max_{1 \leq i \leq n} \left\{ D_i^+ + \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2} \right\}, \quad (4.2)$$

and

$$\mu^D(\vec{G}) \leq \max_{1 \leq i \leq n} \left\{ D_i^+ + \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2} \right\}. \quad (4.3)$$

Moreover, if the equality in (4.2) (or (4.3)) holds then $D_i^+ + \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2} = D_j^+ + \sqrt{(n-1) \sum_{k=1}^n d_{kj}^2}$ for any $i, j \in \{1, 2, \dots, n\}$.

Proof. We apply Theorem 2.1 to $\mathcal{Q}(\vec{G})$. Since $b_{ii} = D_i^+$, $b_{ij} = d_{ij} \neq 0$ for all $i \neq j$, $b_{ii} = d_{ii} = 0$, and $l_i = n - 1$ for $i = 1, 2, \dots, n$, then $b_{ii} + \sqrt{\sum_{k \neq i} l_k b_{ki}^2} = D_i^+ + \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2}$ for $i = 1, 2, \dots, n$, and thus (4.2) holds by (2.1). By Corollary 2.5 and (i), (4.3) holds.

It is easy that if the equality in (4.2) (or (4.3)) holds then $D_i^+ + \sqrt{(n-1) \sum_{k=1}^n d_{ki}^2} =$

$D_j^+ + \sqrt{(n-1) \sum_{k=1}^n d_{kj}^2}$ for any $i, j \in \{1, 2, \dots, n\}$ by Theorem 2.1. □

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